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QUASI- AND WEAKLY-QUASI-FIRST-COUNTABLE SPACES

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In this paper, we study two countability properties that are weaker than the first-countability property and following the pattern of Michael [3], we obtain the spaces with such properties as quotient images of metric spaces by particular kinds of maps.

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In studying the notion of first-countability, one often gives as an easy example of a non-first-countable space, the quotient obtained from the disjoint union of $\aleph_0$ copies of a convergent sequence, say $\{1/n;\ n\in\mathbb{N}\}\cup\{0\}$, by identifying all copies of 0 to a point x_0. However, the way in which this space fails to be first-countable is not too drastic because the space still carries some notion of countability: indeed, at x_0, there are countably many branches on each of which we can fix countably many sets (namely the tails of the sequences) such that to construct a neighborhood of x_0, it suffices to pick one set on each branch. This countability property, which we are going to call quasi-first-countability, is not shared by all non-first-countable spaces, not even by countable non-first-countable spaces. Indeed, in [5], we showed that some countable sequential spaces (for details about sequential spaces, see [1] and [2]) were quotient of countable metric spaces, while some others were not. The notion of quasi-first-countability and, as we will see below, of weakly-quasi-first-countability, came up as we were trying to identify which countable sequential spaces were quotients of a countable metric space, by characterizing internally the topology of such spaces. Theorems 2′ and 5 characterize those spaces as particular quotient images of metric spaces.

1. Quasi-first-countable spaces.

Definition. We say that a space X is *quasi-first-countable at* $x_0 \in X$ if there exist countably many countable families of decreasing subsets of X containing x_0 such that a subset V of X is a neighborhood of x_0 in X if and only if V contains a member of each family.

We say that X is *quasi-first-countable* if it is so at each of its points.

The notion of quasi-first-countability retains enough of the idea of first-countability to insure "sequentialness" and in fact "Fréchetness", that is:

Proposition. *Every quasi-first-countable space is Fréchet.*

Proof. Let $x_0 \in \bar{A}$ and let $(B^n_m)_{m\in\mathbf{N}}$ be the families provided by quasi-first-countability at x_0. There exists $n_0 \in \mathbf{N}$ such that each $B^{n_0}_m$ meets A; for otherwise, for any $n \in \mathbf{N}$, we could find $\sigma(n) \in \mathbf{N}$ such that

$$B^n_{\sigma(n)} \cap A = \emptyset.$$

Now by quasi-first-countability $\bigcup_{n\in\mathbf{N}} B^n_{\sigma(n)}$ is a neighborhood of x_0 and $(\bigcup_{n\in\mathbf{N}} B^n_{\sigma(n)}) \cap A = \emptyset$ contradicting the fact that $x_0 \in \bar{A}$. Hence there exists n_0 such that $B^{n_0}_m \cap A \neq \emptyset$ for all $m \in N$; let $x_m \in B^{n_0}_m \cap A$. Then $(x_m)_{m\in\mathbf{N}}$ converges to x_0 since any neighborhood of x_0 contains one of the $B^{n_0}_m$'s. Therefore the space is Fréchet. □

Quasi-first-countability at x_0 is an intermediate property between having $\chi(x_0, X) = \aleph_0$ and $\chi(x_0, X) = c$, where $\chi(x_0, X)$ is the smallest cardinality for a neighborhood base at x_0. Not every space with $\chi(x_0, X) = c$ is quasi-first-countable. In fact, there are countable spaces which are not quasi-first-countable. One such example is the space Q^*_p constructed in [5] which is the quotient of the following subset of $\mathbf{R}^2$:

$$Q \times (Q - \{0\}) \cup P \times \{0\}$$

(Q is the set of rational numbers and P the set of irrational numbers) obtained by identifying $P \times \{0\}$ to a single point.

As we said in the introduction, the notion of quasi-first-countability came up as we were trying to identify which countable sequential spaces were quotients of a countable metric space by characterizing internally the topology of such spaces. Theorem 1 and Theorem 4 below give the answer to that problem.

Theorem 1. *A countable space X is quasi-first-countable if and only if it is a hereditarily quotient image of a countable metric space.*

Proof. Suppose X is countable and quasi-first-countable. For each x, let $(B^{x,n}_m)_{m\in\mathbf{N}}$ be the families provided by quasi-first countability at x. Let $Y^{x,n}$ be the set X

provided with the discrete topology except for the point x, which has as a base of neighborhoods the family $(B_m^{x,n})_{m\in\mathbf{N}}$. Then $Y^{x,n}$ is first-countable, countable and regular, and hence metric.

Let Y be the disjoint union of all $Y^{x,n}$'s for $x\in X$ and $n\in N$. Then Y is a countable metric space.

Let f be the natural map of Y onto X (mapping a point onto itself). Then f is continuous: for, let 0 be open in X; let $f^{-1}(0)\cap Y^{x,n}\neq\emptyset$. If $x\in 0$, then since 0 is open, 0 contains one of the $B_m^{x,n}$'s for some $m\in\mathbf{N}$; hence $f^{-1}(0)$ contains a neighborhood of x in $Y^{x,n}$ and clearly it contains a neighborhood of each of its other points in $Y^{x,n}$. Hence $f^{-1}(0)\cap Y^{x,n}$ is open and hence f is continuous.

The map f is also a quotient map: for suppose $f^{-1}(0)$ is open in Y, that is $f^{-1}(0)\cap Y^{x,n}$ is open for all $x\in X$ and $n\in\mathbf{N}$. If $x\in 0$, then since $f^{-1}(0)\cap Y^{x,n}$ is open, $f^{-1}(0)\cap Y^{x,n}$ contains a $B_m^{x,n}$ for a certain m; hence 0 contains $B_m^{x,n}$; this is true for each n and hence, by quasi-first-countability, 0 is a neighborhood of x. Therefore, 0 is a neighborhood of each of its points, that is, 0 is open. It follows that f is a quotient map and since X is Hausdorff and Fréchet (by quasi-first-countability), f is hereditarily quotient (see [2]).

Now for the converse, let us assume that X is a hereditarily quotient image of a countable metric space M; let $f:M\to X$ be a quotient map. Let $f^{-1}(x)=\{x_1,x_2,\dots,x_n\dots\}$. We want to show that X is quasi-first-countable at x; let $(C_m^n)_{m=1}^{\infty}$ be a countable base of neighborhoods of M at x_n; let

$$B_m^n=f(C_m^n).$$

Then the $(B_m^n)_{m\in\mathbf{N}}$ are the required families. For, if 0 is a neighborhood of x, then for any n, there exists $\sigma(n)\in\mathbf{N}$ such that

$$C_{\sigma(n)}^n\subset f^{-1}(0)$$

and then $0=f\circ f^{-1}(0)$ contains $B_{\sigma(n)}^n$ for any $n\in\mathbf{N}$. Conversely, if 0 contains some $B_{\sigma(n)}^n$ for all $n\in\mathbf{N}$, then $f^{-1}(0)$ contains $C_{\sigma(n)}^n$ for all n and hence $f^{-1}(0)$ is a neighborhood of $f^{-1}(x)$. This implies that 0 is a neighborhood of x since f is hereditarily quotient. □

Remark. The result cannot be improved, in the sense that "hereditarily quotient" cannot be replaced by "quotient". Indeed let $f:Q\to Q$ be the following map:

$$f(x)=x \quad \text{if } x\neq n \text{ for all } n\in\mathbf{N},$$

$$f(n)=1/n.$$

Let $Q^\circ=f(Q)$ and let us consider on Q° the quotient topology induced by f. Then Q° is a countable sequential space, quotient image of Q, but not a Fréchet space and hence, not a quasi-first-countable space.

2. Quasi-first-countable spaces as quotients of metric spaces

In studies of quotient maps, as done by Michael in [3] and Olson in [4], to spaces with a certain countability property is associated a particular kind of map which enables one to obtain the spaces as images of metric spaces by those maps. For example, a space is sequential if and only if it is a quotient image of some metric space and a space is Fréchet if and only if it is a hereditarily quotient image of some metric space. The relation in that case is in fact very strong since a map between a metric space and a Fréchet Hausdorff space has to be hereditarily quotient (see [1]). In this section, we define the kind of map that should be associated with quasi-first-countable spaces, and we show that the relation is almost as strong as the relation between Fréchet spaces and hereditarily quotient maps.

Definition. A map $f:X\to Y$ is said to have *countable frontier* if for any $y\in Y$, $\mathrm{fr}(f^{-1}(y))$ is countable. (We recall that the frontier of a subset A of a space X is defined as the closure of A in X minus its interior.)

Theorem 2. *A space X is quasi-first-countable if and only if it is the image of a metric space by a hereditarily quotient map that has countable frontier.*

Proof. Suppose X is quasi-first-countable and let $(B_m^{x,n})_{m\in\mathbf{N}}$ be the families provided by this property. Let $Y^{x,n}$ be the set X with the topology in which points are open except for x which has as a base of neighborhoods the family $(B_m^{x,n})_{m\in\mathbf{N}}$. $Y^{x,n}$ is a metric space (for example the following metric is easily seen to be compatible with the topology of $Y^{x,n}$:

$$d(y,x)=1/m \quad \text{where } m \text{ is the smallest integer such that } y\in B_m^{x,n},$$

$$d(y_1,y_2)=d(y_1,x)+d(x,y_2) \quad \text{for } y_1\neq x \text{ and } y_2\neq x).$$

We define the space Y to be the disjoint union of all $Y^{x,n}$'s and we consider the natural map $f:Y\to X$. Arguments similar to those used in the proof of Theorem 1 show that f is a quotient map and hence hereditarily quotient since X is Fréchet. Furthermore, $\mathrm{fr}(f^{-1}(x))$ is countable since $\mathrm{fr}(f^{-1}(x))$ is formed by the point x of each $Y^{x,n}$ as n runs through $\mathbf{N}$.

Conversely, let $f:M\to X$ be a hereditarily quotient map that has countable frontier, with M a metric space. Let $\mathrm{fr}(f^{-1}(x))=\{x_1,x_2,\ldots,x_n,\ldots\}$. Let $(C_m^{x,n})_{m\in\mathbf{N}}$ be a neighborhood base at x_n and let

$$B_m^{x,n}=f(C_m^{x,n}).$$

Again an argument as in Theorem 1 shows that the families $(B_m^{x,n})_{m\in\mathbf{N}}$ are as required. □

A quotient map between a metric space and a quasi-first-countable space does not have to have countable frontier. In fact, in a metric space, as far as neighborhoods are concerned, compact sets and points behave the same way (more precisely, one can show that compact subsets have countable character in metric spaces), and hence, one sees easily that in the second part of the proof above, "countable frontier" could be replaced by "σ-compact frontier" (that is $\operatorname{fr}(f^{-1}(x))$ is σ-compact for each $x \in X$). Hence, we also have:

Theorem 2'. *A space X is quasi-first-countable if and only if it is the image of a metric space by a hereditarily quotient map that has σ-compact frontier.*

Still, a quotient map between a metric space and a quasi-first-countable space need not have σ-compact frontier: for example, take X to be a metric space, $M(X)$ to be the disjoint union of uncountably many copies of X and f to be the natural map of $M(X)$ onto X (mapping points onto themselves). However, we do have a result in that direction for quotients that do not involve too many identifications. Namely, let us call a quotient map $f: Y \to X$ a "*nice quotient*" if whenever $x \in X$ is such that $f^{-1}(x)$ contains more than one point, there is a neighborhood V of x in X such that x is the only element of V with an inverse image of more than one point, that is the elements x of X for which $f^{-1}(x)$ contains more than one point form a relatively discrete subspace of X. Then we have the following:

Theorem 3. *Let $f: M \to X$ be a "nice quotient map" of a metric space onto a quasi-first-countable space. Then f is hereditarily quotient and has σ-compact frontier.*

Proof. Since X is Hausdorff and Fréchet the map f is hereditarily quotient. Let $x \in X$ be such that $f^{-1}(x)$ contains more than one element. We want to show that $f^{-1}(x)$ has σ-compact frontier.

Let $(B^n_m)_{m \in \mathbf{N}}$ be the families guaranteed by quasi-first-countability at x; we may assume that $B^n_m \subset V$ for each $n, m \in \mathbf{N}$. For each $y \in \operatorname{fr}(f^{-1}(x))$, we can find a sequence $(y_n)_{n \in \mathbf{N}}$ in $M - f^{-1}(x)$ such that:

$$y_n \to y$$

and

$$\text{for some } n_0 \in \mathbf{N},\ B^{n_0}_m \text{ contains a tail of } (f(y_n))_{n \in \mathbf{N}} \text{ for all } m \in \mathbf{N}.$$

Indeed, let $(z_n)_{n \in \mathbf{N}}$ be any sequence of $M - f^{-1}(x)$ converging to y; then $(f(z_n))_{n \in \mathbf{N}}$ converges to x; now there exists $n_0 \in \mathbf{N}$ such that, for all $m \in \mathbf{N}$,

$$B^{n_0}_m \cap \{f(z_n);\ n \in \mathbf{N}\} \neq \emptyset.$$

For otherwise, for each $n \in \mathbf{N}$ we could find $\sigma(n) \in \mathbf{N}$ such that

$$B^n_{\sigma(n)} \cap \{f(z_n);\ n \in \mathbf{N}\} = \emptyset$$

and then $\bigcup_{n \in \mathbf{N}} B^n_{\sigma(n)}$ would be a neighborhood of x not meeting $(f(z_n))_{n \in \mathbf{N}}$ contradicting the fact that $(f(z_n))_{n \in \mathbf{N}}$ converges to x. Hence there does exist n_0 such that

$$B_m^{n_0} \cap \{f(z_n);\, n \in \mathbb{N}\} \neq \emptyset$$

for all $m \in \mathbb{N}$; using this and the fact that the $B_m^{n_0}$'s are decreasing, one can construct by induction a subsequence $(z_{\sigma(n)})_{n\in\mathbb{N}}$ such that

$$f(z_{\sigma(n)}) \in B_m^{n_0}$$

Let $y_n = z_{\sigma(n)}$. Then $(y_n)_{n\in\mathbb{N}}$ has the required property, that is each $B_m^{n_0}$ contains a tail of $(f(y_n))_{n\in\mathbb{N}}$.

Now, we associate each $y \in \mathrm{fr}(f^{-1}(x))$ (via this sequence $(y_n)_{n\in\mathbb{N}}$) to the n_0 obtained as above. Let A_{n_0} be the set of y's which are associated to n_0. We claim that A_{n_0} is compact, which will show that $\mathrm{fr}(f^{-1}(x))$ is σ-compact.

Since M is metric, it suffices to show that each sequence in A_{n_0} has an accumulation point in $\mathrm{fr}(f^{-1}(x))$. Let $(y^p)_{p\in\mathbb{N}}$ be a sequence in A_{n_0}. For each y^p, let $(y_n^p)_{n\in\mathbb{N}}$ be the sequence considered above, so that

$$(y_n^p) \to y^p$$

and

$$B_m^{n_0} \text{ contains a tail of } (f(y_n^p))_{n\in\mathbb{N}} \text{ for any } m.$$

Since $B_m^{n_0}$ contains a tail of $(f(y_n^m))_{n\in\mathbb{N}}$, we can find $z_m \in (y_n^m)_{n\in\mathbb{N}}$ such that

$$d(y^m, z_m) < 1/m \quad \text{and} \quad f(z_m) \in B_m^{n_0}.$$

We then get a sequence $(z_m)_{m\in\mathbb{N}}$ such that $(f(z_m))_{m\in\mathbb{N}}$ converges to x since any neighborhood of x contains $B_m^{n_0}$ for some m and each $B_p^{n_0}$ contains all $f(z_m)$ for $m \geq p$. Since $(f(z_m))_{m\in\mathbb{N}}$ converges to x and f is hereditarily quotient, replacing $(f(z_m))_{m\in\mathbb{N}}$ by a subsequence if necessary, we can say that there is a sequence $(y_n)_{n\in\mathbb{N}}$ in M and a point $y \in f^{-1}(x)$ such that

$$y_n \to y \quad \text{and} \quad f(y_n) = f(z_n).$$

Now, since the $f(z_m)$'s are in the sets $B_m^{n_0}$ and hence in V, then we must have $y_n = z_n$ for all $n \in \mathbb{N}$; but $y_n \to y$; hence $z_n \to y$. Furthermore, we have $d(z_p, y^p) < 1/p$; hence we can conclude that $y^p \to y$. Therefore the sequence $(y^p)_{p\in\mathbb{N}}$ has an accumulation point in $\mathrm{fr}(f^{-1}(x))$ and this proves that $\bar{A}_{n_0}$ is compact. □

3. Weakly-quasi-first-countable spaces

We identified in Theorem 1 the countable sequential spaces that are hereditarily quotient images of countable metric spaces. We want to do the same now for countable sequential spaces that are simply quotients of countable metric spaces.

Definition. A space X is said to be *weakly-quasi-first-countable* if and only if for all $x \in X$, there exist countably many countable families of decreasing subsets containing

x such that a set 0 is open if and only if for any $x \in 0$, 0 contains a member of each family associated to x.

A straightforward argument shows that every weakly-quasi-first-countable space is sequential since if a set contains a tail of any sequence converging to one of its points, it must also contain a member of each family associated to its points; but weakly-quasi-first-countable spaces need not be Fréchet as is shown by the space given in the remark at the end of Section 1.

Weak quasi-first-countability is the property we were looking for in order to characterize internally by their topologies the quotients of countable metric spaces.

Theorem 4. *A countable space X is a quotient of a countable metric space if and only if it is weakly-quasi-first-countable.*

Proof. The proof can be copied on the proof of Theorem 1 doing the few necessary changes.

The weakly-quasi-first-countable spaces can be characterized as was done for quasi-first-countable spaces and the proof of Theorem 2 can easily be adapted to prove the following theorem:

Theorem 5. *A space X is weakly-quasi-first-countable if and only if it is the image of a metric space by a quotient map with countable frontier (or σ-compact frontier as in Theorem 2′).*

We have defined the notion of weak quasi-first-countability in order to characterize internally the quotients of countable metric spaces and in doing so, we were forced to define it as a global property instead of as a local property as for quasi-first-countability (that is we could not find a way to define it at a point in such a way that if it is satisfied at each point, then it is equivalent to the definition we gave). This reflects the fact that "sequentialness" is defined as a global property while "Fréchetness" can be considered a local property (we could say that X is "Fréchet at x" if $x \in \bar{A} \rightarrow$ there exists a sequence in A converging to x, and X is Fréchet if it is so at each of its points). The difference between the definitions of weak quasi-first-countability and quasi-first-countability is the exact translation of the difference between sequential spaces and quotient maps on one hand, Fréchet spaces and hereditarily quotient maps on the other hand. It is the difference between topologies given in terms of open sets and topologies given in terms of neighborhoods. Indeed, quotient maps are defined by

"$f^{-1}(U)$ is open if and only if U is open" while hereditarily quotient maps can be equivalently defined by

"$f^{-1}(U)$ is a neighborhood of $f^{-1}(x)$ if and only if
U is a neighborhood of x".

Similarly, sequential spaces are defined by

"sequentially open sets are open",

while Fréchet spaces, as one can easily check, using nets, could be equivalently defined by

"sequential neighborhoods of x are neighborhoods of x",

where one defines a sequential neighborhood of x in a natural way as a set containing a tail of each of the sequences converging to x.

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